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The solution of some dual equations with an application in the theory of elasticity

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SUMMARY

In this paper the author solves some dual equations involving an inverse Mellin type transform. The use of these equations is then illustrated by their application to a crack problem in the theory of elasticity.

1. The dual equations

The object of this paper is to find the solution of some dual equations involving the inverse of the Mellin type transform H_R which is defined by the equation

$$H_R[f(r); s] = \int_R^\infty [r^{s-1} + R^{2s} r^{-s-1}] f(r) dr , \tag{1.1}$$

and which was first introduced by D. Naylor in his paper [1].

The equations to be solved are

$$\begin{aligned} H_R^{-1}\left[A(s)\tan\frac{\pi s}{n}\,;\, r\right] &= f(r), \quad R < r < a \\ H_R^{-1}[s^{-1}A(s);\quad r] &= 0, \qquad a < r < \infty \end{aligned} \tag{1.2}$$

where $|\mathrm{Re}(s)| < \frac{1}{2}n$, and the method of solution, which is similar to the "elementary method" of Sneddon [2], depends on the assumption that $A(s)$ may be written in the form

$$A(s) = \int_R^a t^{\frac{1}{2}n-1} p(t^n)(t^s - R^{2s} t^{-s}) dt . \tag{1.3}$$

With this choice of $A(s)$ we find, on making use of the result

$$H_R[H(t-r); s] = s^{-1}(t^s - R^{2s} t^{-s}), \qquad |\mathrm{Re}(s)| < \infty ,$$

that

$$H_R^{-1}[s^{-1}A(s); r] = \begin{cases} \int_r^a t^{\frac{1}{2}n-1} p(t^n) dt , & R < r < a \\ 0, & a < r < \infty \end{cases} \tag{1.4}$$

and hence that the second dual equation is satisfied automatically.

Similarly, on substituting from (1.3) into the first dual equation and making use of the result

$$H_R\left[\frac{(rt/R^2)^{\frac{1}{2}n}}{1-(rt/R^2)^n} - \frac{(rt)^{\frac{1}{2}n}}{t^n - r^n}\,;\, s\right] = \frac{\pi}{n}(t^s - R^{2s}t^{-s})\tan(\pi s/n) ,$$

$$R < t < \infty , \quad |\mathrm{Re}(s)| < \tfrac{1}{2}n ,$$

we find that it too will be satisfied if

$$\frac{n}{\pi}\int_R^a t^{n-1} p(t^n)\left\{\frac{R^n}{R^{2n} - r^n t^n} - \frac{1}{t^n - r^n}\right\} dt = r^{-\frac{1}{2}n} f(r), \qquad a < r < R . \tag{1.5}$$

If we now let $\tau = t^n/R^n$, $b = a^n/R^n$ and then put ρ equal to r^n/R^n and R^n/r^n in turn, we find that (1.5) is equivalent to the equation

$$\frac{1}{\pi}\int_{b^{-1}}^{b} \frac{h(\tau)}{\tau-\rho}\,d\tau = g(\rho)\,, \qquad b^{-1} < \rho < b \tag{1.6}$$

where

$$h(\tau) = \left| \begin{array}{ll} -R^{\frac{1}{2}n}p(R^n\tau)\,, & 1<\tau<b \\ \tau^{-1}R^{\frac{1}{2}n}p(R^n\tau^{-1})\,, & b^{-1}<\tau<1 \end{array} \right. \tag{1.7}$$

and

$$g(\rho) = \left| \begin{array}{ll} \rho^{-\frac{1}{2}}f(R\rho^{1/n})\,, & 1<\rho<b \\ \rho^{-\frac{1}{2}}f(R\rho^{-1/n})\,, & b^{-1}<\rho<1\,. \end{array} \right. \tag{1.8}$$

Now equation (1.6) is well known and its solution (see [3]) is given by

$$h(\tau) = \frac{1}{\Delta(\tau)}\left\{C - \frac{1}{\pi}\int_{b^{-1}}^{b} \frac{\Delta(y)g(y)}{y-\tau}\,dy\right\}, \tag{1.9}$$

where $\Delta(y) = [(b-y)(y-b^{-1})]^{\frac{1}{2}}$ and C is an arbitrary constant. For the case on hand however, $h(\tau^{-1}) = -\tau h(\tau)$ and $g(\rho^{-1}) = \rho g(\rho)$ and hence we find that

$$C = \frac{1}{2\pi}\int_{b^{-1}}^{b} y^{-1}\Delta(y)g(y)dy\,. \tag{1.10}$$

It now follows that

$$h(\tau) = \frac{1}{2\pi\Delta(\tau)}\int_{b^{-1}}^{b}\left(\frac{\tau+y}{\tau-y}\right)\frac{\Delta(y)g(y)}{y}\,dy \tag{1.11}$$

and therefore, on changing back to the original variables that

$$p(t^n) = \frac{n(R^{2n}-t^{2n})}{\pi[(a^n-t^n)(a^nt^n-R^{2n})]^{\frac{1}{2}}}\int_R^a \frac{[(a^n-y^n)(a^ny^n-R^{2n})]^{\frac{1}{2}}}{(t^n-y^n)(t^ny^n-R^{2n})} f(y)\,y^{\frac{1}{2}n-1}dy\,. \tag{1.12}$$

The result is now obtained by substituting from (1.12) into (1.3).

2. An application in the theory of elasticity

In order to illustrate the use of the equations investigated above we shall now consider the problem of determining the stress intensity factor and the crack energy of a pair of cracks which originate at the edge of a circular hole in an infinite elastic solid under longitudinal shear. We shall assume that the cracks and the hole are traction free and that, in cylindrical coordinates (r, ϑ, z), they are defined by the relations $R \leqq r \leqq Rb$, $\vartheta = 0, \pi$, $-\infty < z < \infty$ and $0 \leqq r \leqq R$, $0 \leqq \vartheta \leqq 2\pi$, $-\infty < z < \infty$ respectively. We assume also that as r tends to infinity, σ_{rz} tends to $T\sin\vartheta$ and $\sigma_{\vartheta z}$ to $T\cos\vartheta$ (Fig. 1).

In the longitudinal shear problem, the fields of displacement and stress in the body under consideration are such that

$$\begin{aligned} &u_r = u_\vartheta = 0\,, \quad u_z = w(r,\vartheta) \\ &\sigma_{rr} = \sigma_{\vartheta\vartheta} = \sigma_{zz} = \sigma_{r\vartheta} = 0\,, \\ &\sigma_{\vartheta z} = \frac{\mu}{r}\frac{\partial w}{\partial \vartheta} \quad \text{and} \quad \sigma_{rz} = \mu\frac{\partial w}{\partial r} \end{aligned} \tag{2.1}$$

where μ is the shear modulus and w is a solution of Laplace's equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2} = 0\,. \tag{2.2}$$

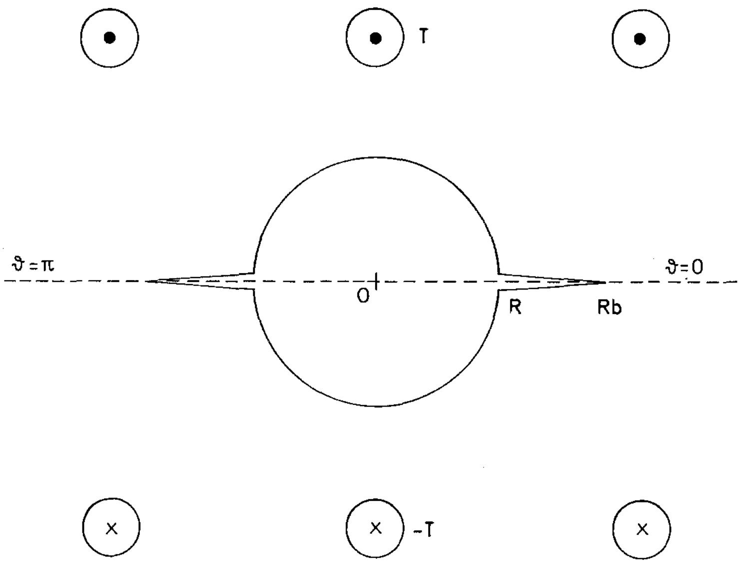

Figure 1.

Because of the symmetries of the problem it will be sufficient to find a function $w(r, \vartheta)$ which satisfies equation (2.2) in the region $R < r < \infty$, $0 < \vartheta < \frac{1}{2}\pi$ and which is such that

(1) As r tends to infinity, $r^{-1}(\partial w/\partial \vartheta)$ tends to $T/\mu \cos \vartheta$ and

$$\frac{\partial w}{\partial r} \text{ to } \frac{T}{\mu} \sin \vartheta ,$$

(2) $\dfrac{\partial w}{\partial r}(R, \vartheta) = 0, \quad 0 < \vartheta < \frac{1}{2}\pi ,$

(3) $\dfrac{\partial w}{\partial \vartheta}(r, \frac{1}{2}\pi) = 0, \quad R \leqq r < \infty ,$

(4) $w(r, 0) \quad = 0, \quad Rb \leqq r < \infty ,$

and

(5) $\dfrac{\partial w}{\partial \vartheta}(r, 0) \quad = 0, \quad R < r < Rb .$

Let $w_1(r, \vartheta)$ be a solution of (2.2) in the given region. Let it satisfy conditions (2) and (3) and be $O(r^{-1-\alpha})$ at infinity for some $\alpha > 0$. Then, on applying the transform H_R to the equation (2.2) and making use of the result

$$H_R\left[r\frac{\partial}{\partial r} r\frac{\partial}{\partial r} w_1(r, \vartheta); s\right] = s^2 \bar{w}_1(s, \vartheta) - 2R^{s+1}\frac{\partial w_1}{\partial r}(R, \vartheta)$$

where

$$\bar{w}_1(s, \vartheta) = H_R[w_1(r, \vartheta); r \rightarrow s], \quad |\mathrm{Re}(s)| < 1 \tag{2.3}$$

we find that

$$\frac{d^2 \bar{w}_1}{d\vartheta^2} + s^2 \bar{w}_1 = 0 . \tag{2.4}$$

It follows that

$$\bar{w}_1(s,\vartheta) = s^{-1}[A(s)\cos s\vartheta + B(s)\sin s\vartheta] \tag{2.5}$$

where $A(s)$ and $B(s)$ are arbitrary functions of s. From condition (3) however, we see that

$$\bar{w}_1(s,\vartheta) = \frac{A(s)}{s}\frac{\cos(\vartheta-\frac{1}{2}\pi)s}{\cos\frac{1}{2}\pi s} \tag{2.6}$$

and hence that

$$w_1(r,\vartheta) = H_R^{-1}\left[\frac{A(s)}{s}\frac{\cos(\vartheta-\frac{1}{2}\pi)s}{\cos\frac{1}{2}\pi s}\,;\,r\right], \quad |\mathrm{Re}(s)|<1\,. \tag{2.7}$$

It is now a simple matter to show that the function

$$w(r,\vartheta) = \frac{T}{\mu}(r+R^2r^{-1})\sin\vartheta + H_R^{-1}\left[\frac{A(s)}{s}\frac{\cos(\vartheta-\frac{1}{2}\pi)s}{\cos\frac{1}{2}\pi s}\,;\,r\right], \tag{2.8}$$

$|\mathrm{Re}(s)|<1$, is a solution of equation (2.2) in the region $R<r<\infty$, $0\leqq\vartheta\leqq\frac{1}{2}\pi$, and that it satisfies conditions (1), (2) and (3). If we now apply conditions (4) and (5) we find that $A(s)$ must satisfy the dual equations

$$H_R^{-1}[A(s)\tan\tfrac{1}{2}\pi s\,;\,r] = -\frac{T}{\mu}(r+R^2r^{-1}), \quad R<r<Rb$$

$$H_R^{-1}[s^{-1}A(s)\,;\,r] = 0\,, \quad Rb<r<\infty \tag{2.9}$$

$|\mathrm{Re}(s)|<1$ and hence, by the results of section 1, that

$$A(s) = \int_R^{Rb} p(t^2)(t^s - R^{2s}t^{-s})dt \tag{2.10}$$

where

$$p(t^2) = \frac{2T(t^4-R^4)}{\pi\mu[(R^2b^2-t^2)(b^2t^2-R^2)]^{\frac{1}{2}}} \times \int_R^{Rb}\frac{[(R^2b^2-y^2)(b^2y^2-R^2)]^{\frac{1}{2}}}{(t^2-y^2)(t^2y^2-R^4)}(y+R^2y^{-1})dy\,. \tag{2.11}$$

Furthermore, from (1.4), (2.1) and (2.8) we see that

$$u_z(r,0) = \int_r^{Rb} p(t^2)dt\,, \quad R<r<Rb\,. \tag{2.12}$$

We shall now calculate the stress intensity factor K and the total crack energy W which are defined by the equations

$$K = -\mu \lim_{r\to Rb_-}[2(Rb-r)]^{\frac{1}{2}}\frac{\partial u_z}{\partial r}(r,0) \tag{2.13}$$

and

$$W = 2T\int_R^{Rb}(1+R^2r^{-2})u_z(r,0)dr \tag{2.14}$$

respectively.

On substituting from (2.12) into (2.13) we see that

$$K = \mu \lim_{r\to Rb_-}[2(Rb-r)]^{\frac{1}{2}}p(r^2)$$

and hence by (2.11) that

$$K = \frac{2T}{\pi}\left(\frac{R(b^4-1)}{b}\right)^{\frac{1}{2}}\int_R^{Rb}\frac{(y+R^2y^{-1})dy}{[(R^2b^2-y^2)(b^2y^2-R^2)]^{\frac{1}{2}}}$$

$$= Tb^{-\frac{3}{2}}[R(b^4-1)]^{\frac{1}{2}}\,. \tag{2.15}$$

Similarly, if we substitute from (2.12) into (2.14) we find that

$$W=\frac{4T^2}{\pi\mu}\int_R^{Rb}\frac{(t^4-R^4)(t-R^2t^{-1})dt}{[(R^2b^2-t^2)(b^2t^2-R^2)]^{\frac{1}{2}}}\int_R^{Rb}\frac{[(R^2b^2-y^2)(b^2y^2-R^2)]^{\frac{1}{2}}}{(t^2-y^2)(t^2y^2-R^4)}\left(y+\frac{R^2}{y}\right)dy$$

$$=\frac{\pi T^2R^2(b^2-1)^2}{2\mu b^2}\,.$$

REFERENCES

[1] D. Naylor, On a Mellin type integral transform, *Journ. Math. and Mech.*, 12 (1963) 265–274.
[2] I. N. Sneddon, The elementary solution of dual integral equations, *Proc. Glas. Math. Assoc.*, 4 (1960) 108–110.
[3] F. A. Tricomi, On the finite Hilbert transformation, *Quart. Journ. Math.*, 2 (1951) 199.